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Astrogeometry: Toward Mathematical Foundations

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In *image processing* (e.g., in *astronomy*), the desired black-and-white image is, from the mathematical viewpoint, a *set*. Hence, to process images, we need to process sets. To define a generic set, we need infinitely many parameters; therefore, if we want to represent and process sets in the computer, we must restrict ourselves to finite-parameter families of sets that will be used to approximate the desired sets. The wrong choice of a family can lead to longer computations and worse approximation. Hence, it is desirable to find the family that it is *the best* in some reasonable sense. In this paper, we show how the problems of choosing the optimal family of sets can be formalized and solved. As a result of the described general methodology, for *astronomical images*, we get exactly the geometric shapes that have been empirically used by astronomers and astrophysicists; thus, we have a *theoretical explanation* for these shapes.

1. INTRODUCTION TO THE PROBLEM

1.1. Sets Are Needed for Image Processing

In *image processing*, our goal is to restore the actual image. For black-and-white images, the image can be identified with a set of its black points, i.e., with a set in a 2D or in a 3D space. So, in order to process images, we must be able to process sets.

1.2. In the Computer, We Can Only Use Finite-Parameter Families of Sets

Images can, in principle, be arbitrarily complicated.

The ideal description of a set $X \subseteq R^k$ would include, for any point $x \in R^k$, information on whether this point x belongs to the given set X or not.

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This information requires infinitely many *bits* (binary digits) to store. However, inside any given computer, we can only store finitely many bits, and therefore we can represent the information only about finitely many points $x \in R^k$. In computer imaging, these points are usually called *pixels*.

A pixel-by-pixel representation is necessary for some images (e.g., to store a high-quality photo) and for computer games (to create the most realistic picture). However, such a representation requires a lot of computer memory and makes processing the corresponding data extremely slow. Therefore, if we want to speed up the processing of these sets, we must somehow approximate arbitrarily complicated sets by sets that can be characterized by a few real-valued parameters, i.e., by sets that belong to some *finite-dimensional family of sets*.

Several families of this type have been efficiently used in image processing. This leads us to a problem:

1.3. Main Problem: Which Families of Sets Should We Choose?

In principle, different families of sets can be used. It turns out that often, the use of different approximating families leads to different quality of the resulting approximation. Therefore, it is important to choose the right approximating family.

Currently, this choice is mainly made ad hoc, at best, by testing a few possible families and choosing the one that performs the best on a few benchmarks. Since only a few families are analyzed, one is not sure that one did not miss a really good approximating family (and since only a few benchmarks are used for comparison, we are not sure that the chosen family is indeed the best one). It is therefore desirable to find the *optimal* family of approximating sets.

1.4. Goal of This Paper

In this paper, we will describe a general framework for finding the optimal family, and illustrate this general idea on the example of astronomic imaging.

2. WHAT DOES "OPTIMAL" MEAN? MOTIVATIONS FOR THE FOLLOWING DEFINITIONS

2.1. What Is an "Optimality Criterion"?

When we say "optimal," we mean optimal with respect to some *optimality criterion*. When we say that some *optimality criterion* is given, we mean that, given two different families of approximating sets, we can decide whether the first one is better, or the second one is better, or these families are of the same quality with respect to the given criterion. In mathematical terms, this means that we have a preordering relation $\preccurlyeq$ on the set of all possible finite-dimensional families of sets.

2.2. We Want to Solve an Ambitious Problem: Enumerate All Finite-Dimensional Families of Sets That Are Optimal Relative to Some Natural Criteria

One way to approach the problem of choosing the "best" family of sets is to select *one* optimality criterion and to find a family of sets that is the best with respect to this criterion. The main drawback of this approach is that there can be different optimality criteria, and they can lead to different optimal solutions. It is therefore desirable not only to describe a family of sets that is optimal relative to *some* criterion, but to describe *all* families of sets that can be optimal relative to different natural criteria. In this paper, we implement exactly this more ambitious task.

2.3. Examples of Optimality Criteria

2.3.1. Numerical Optimality Criteria

Preordering is the general formulation of optimization problems in general, not only of the problem of choosing a family of sets. In general optimization theory, in which we are comparing arbitrary *alternatives* $A, B, \ldots,$ from a given set $\mathcal{A}$, the most frequent case of such a preordering is when a *numerical criterion* is used, i.e., when a function J: $\mathcal{A} \rightarrow R$ is given for which $A \preccurlyeq B$ iff $J(A) \leq J(B)$.

Several natural numerical criteria can be proposed for choosing the best family of sets: if we approximate the actual set of possible values X by an element $\tilde{X}$ from the chosen family, then we can measure the quality of the approximation by computing the Lebesgue measure of the difference between the two sets or by computing the Hausdorff distance between these two sets. As an optimality criterion, we can, e.g., choose the *average* value of this quality measure (average in the sense of some natural probability measure on the class of all problems).

Alternatively, we can fix a class of the problem and take the *largest* (worst-case) value of the quality measure for problems of this class as the desired (numerical) optimality criterion.

2.3.2. Nonnumerical Optimality Criteria Naturally Appear

For "worst-case" optimality criteria, it often happens that there are several different alternatives that perform equally well in the worst case, but whose performance differs drastically in the average cases. In this case, it makes sense, among all the alternatives with the optimal *worst-case* behavior, to choose the one for which the *average* behavior is the best possible. This very natural idea leads to an optimality criterion that is *not* described by a numerical optimality criterion $J(A)$: in this case, we need *two* functions: $J_1(A)$ describes the worst-case behavior, $J_2(A)$ describes the average-case behavior, and $A \preccurlyeq B$ iff either $J_1(A) < J_2(B)$ or $J_1(A) = J_1(B)$ and $J_2(A) \leq J_2(B)$.

We could further specify the described optimality criterion and end up with *a* natural criterion. However, as we have already mentioned, the goal of this paper is not to find *a* family of sets that is optimal relative to some criterion, but to describe *all* families of sets that are optimal relative to some natural optimality criteria. In view of this goal, in the following we will not specify the criterion, but will describe a very general class of *natural* optimality criteria.

So, let us formulate what "natural" means.

2.4. Which Optimality Criteria Are Natural?

2.4.1. The Criterion Must Be Invariant

Problems related to geometric sets often have natural *symmetries*. For example, let us consider astronomical images. These images are sets in R^3 (or in R^2). For such sets, there are three natural symmetries:

First, if we *change the starting point* of the coordinate system from the previous origin point $O = (0, 0, 0)$ to the new origin O' whose coordinates were initially $a = (a_1\ a_2, a_3)$, then each point x with old coordinates (x_1, x_2, x_3) gets new coordinates $x_i' = x_i - a_i$. As a result, in the new coordinates, each set $X \in A$ from a family of images A will be described by a "shifted" set $T_a(X) = \{x - a \mid x \in X\}$, and the family turns into $T_a(A) = \{T_a(X) \mid X \in A\}$. It is reasonable to require that the relative quality of the two families of sets does not depend on the choice of the origin. In other words, we require that if A is better than B, then the "shifted" A [i.e., $T_a(A)$] should be better than the "shifted" B [i.e., than $T_a(B)$].

Second, the choice of a *rotated* coordinate system is equivalent to rotating all the points $[x \to R(x)]$, i.e., going from a set X to a set $R(X) = \{R(x) \mid x \in X\}$, and from a family A to a new family $R(A) = \{R(X) \mid X \in A\}$. It is natural to require that the optimality criterion is invariant with respect to rotations, i.e., if A is better than B, then $R(A)$ is better than $R(B)$.

Third, it is often difficult to find the exact distance to the observed object. Therefore, we are not sure whether the observed image belongs to a small nearby object or to a larger but distant one. As a result of this uncertainty, the actual image is only known modulo homothety (similarity, dilation) $x \to$

$\lambda \cdot x$ for some real number $\lambda > 0$. It is therefore natural to require that the desired optimality criterion be invariant with respect to homothety.

2.4.2. The Criterion Must Be Final

If the criterion does not select any family as an optimal one, i.e., if, according to this criterion, none of the families is better than the others, then this criterion is of no use in selection.

If the criterion considers several different families equally good, then we can always use some other criterion to help select between these "equally good" ones, thus designing a two-step criterion. If this new criterion still does not select a unique family, we can continue this process until we arrive at a combination multistep criterion for which there is only one optimal family.

Therefore, we can always assume that our criterion is *final* in the sense that it selects one and only one optimal family.

3. DEFINITIONS AND THE MAIN RESULT

Our goal is to choose the best finite-parameter family of sets. To formulate this problem precisely, we must formalize what a finite-parameter family is and what it means for a family to be optimal. In accordance with our informal description, both formalizations will use natural symmetries. So, we will first formulate how symmetries can be defined for families of sets, then what it means for a family of sets to be finite-dimensional, and, finally, how to describe an optimality criterion.

Definition 1. Let $g: M \to M$ be a 1–1 transformation of a set M, and let A be a family of subsets of M. For each set $X \in A$, we define the result $g(X)$ of applying this transformation g to the set X as $\{g(x) | x \in X\}$, and we define the result $g(A)$ of applying the transformation g to the family A as the family $\{g(X) | X \in A\}$.

Definition 2. Let M be a smooth manifold. A group G of transformations $M \to M$ is called a *Lie transformation group* if G is endowed with a structure of a smooth manifold for which the mapping $g, a \to g(a)$ from $G \times M$ to M is smooth.

We want to define r-parameter families sets in such a way that symmetries from G would be computable based on parameters. Formally:

Definition 3. Let M and N be smooth manifolds.

- By a *multivalued function* $F: M \to N$ we mean a function that maps each $m \in M$ into a discrete set $F(m) \subseteq N$.

- We say that a multivalued function is *smooth* if for every point $m_0 \in M$ and for every value $f_0 \in F(m)$, there exists an open neighborhood U of m_0 and a smooth function f: $U \to N$ for which $f(m_0) = f_0$ and for every $m \in U, f(m) \subseteq F(m)$.

Definition 4. Let G be a Lie transformation group on a smooth manifold M.

- We say that a class A of closed subsets of M is *G-invariant* if for every set $X \in A$, and for every transformation $g \in G$, the set $g(X)$ also belongs to the class.
- If A is a G-invariant class, then we say that A is a *finitely parameter family of sets* if there exist:
  —a (finite-dimensional) smooth manifold V;
  —a mapping s that maps each element $v \in V$ into a set $s(v) \subseteq M$; and
  —a smooth multivalued function Π: $G \times V \to V$
  such that:
  —the class of all sets $s(v)$ that corresponds to different $v \in V$ coincides with A, and
  —for every $v \in V$, for every transformation $g \in G$, and for every $\pi \in \Pi(g, v)$, the set $s(\pi)$ (which corresponds to π) is equal to the result $g(s(v))$ of applying the transformation g to the set $s(v)$ (which corresponds to v).
- Let $r > 0$ be an integer. We say that a class of sets B is an *r-parameter* class of sets if there exists a finite-dimensional family of sets A defined by a triple (V, s, Π) for which B consists of all the sets $s(v)$ with v from some r-dimensional submanifold $W \subseteq V$.

Definition 5. Let $\mathcal{A}$ be a set, and let G be a group of transformations defined on $\mathcal{A}$.

- By an optimality criterion, we mean a *preordering* (i.e., a transitive reflexive relation) $\preccurlyeq$ on the set $\mathcal{A}$.
- An optimality criterion is called *G-invariant* if for all $g \in G$ and for all $A, B \in \mathcal{A}$, $A \preccurlyeq B$ implies $g(A) \preccurlyeq g(B)$.
- An optimality criterion is called *final* if there exists one and only one element $A \in \mathcal{A}$ that is preferable to all the others, i.e., for which $B \preccurlyeq A$ for all $B \neq A$.
- An optimality criterion is called *natural* if it is G-invariant and final.

Theorem. Let M be a manifold, let G be a d-dimensional Lie transformation group on M, and let $\preccurlyeq$ be a natural (i.e., G-invariant and final) optimality criterion on the class $\mathcal{A}$ of all r-parameter families of sets from M, $r < d$. Then:

- The optimal family A_{opt} is G-invariant; and

- each set X from the optimal family is a union of orbits of $\geq(d - r)$-dimensional subgroups of the group G.

(For the reader's convenience, the proof is given in the final section.)

4. ASTROGEOMETRY: PHYSICAL APPLICATION OF THE MAIN RESULT

Celestial bodies such as galaxies, stellar clusters, planetary systems, etc., have different geometric shapes (e.g., galaxies can be spiral or circular, etc.). Usually, complicated physical theories are used to explain these shapes; for example, several dozen different theories explain why many galaxies are of spiral shape (see, e.g., Toomre and Toomre, 1973; Strom and Strom, 1979; Vorontsov-Veliaminov, 1987; Binney, 1989). Some rare shapes are still unexplained.

In this section, we show that to explain these "astroshapes" we do not need to know the details of *physical* equations: practically all the shapes of celestial bodies can be explained by simple *geometric* invariance properties. This fact explains, e.g., why so many different physical theories lead to the same spiral galaxy shape.

4.1. The Symmetry Group That Corresponds to Astrogeometry

In *astrogeometry* (i.e., in analysis of geometric astronomical images), we are interested in images $X \subset R^3$. As already mentioned, for astronomical images, the natural group of symmetries G_a is generated by shifts, rotations, and dilations.

So, to apply our main result to astrogeometry, we must describe all orbits of subgroups of G_a.

4.2. How to Describe Orbits of Subgroups of G_a

A 1D orbit is an orbit of a 1D subgroup. This subgroup is uniquely determined by its "infinitesimal" element, i.e., by the corresponding element of the Lie algebra of the group G. This Lie algebra is easy to describe. For each of its elements, the corresponding differential equation (which describes the orbit) is reasonably easy to solve.

Two-dimensional forms are orbits of $\geq$ 2D subgroups, so they can be enumerated by combining two 1D subgroups.

Comment. An alternative (slightly more geometric) way of describing 1D orbits is to take into consideration that an orbit, just like any other curve in a 3D space, is uniquely determined by its curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$, where s is the arc length measured from some fixed point. The fact that this curve is an orbit of a 1D group means that for every two points x and x' on this curve, there exists a transformation $g \in G$ that maps x into x'. Shifts and rotations do not change κ_i, they may only shift s (to $s + s_0$); dilations also change s, to $s \to \lambda \cdot s$, and change the numerical values of κ_i. So, for every s, there exist $\lambda(s)$ and $s_0(s)$ such that the corresponding transformation turns a point corresponding to $s = 0$ into a point corresponding to s. As a result, we get functional equations that combine the two functions $\kappa_i(s)$ and these two functions $\lambda(s)$ and $s_0(s)$. Taking an infinitesimal value s in these functional equations, we get differential equations, whose solution leads to the desired 1D orbits.

4.3. As a Result of Applying Our Main Idea, We Get Exactly All Observable Astroshapes

4.3.1. Possible Orbits

The resulting description of 0-, 1-, and 2-dimensional orbits of connected subgroups G_a of the group G is as follows:

0: The only 0-dimensional orbit is a *point*.

1: A generic 1-dimensional orbit is a *conic spiral* that is described (in cylindrical coordinates) by the equations $z = k\rho$ and $\rho = R_0 \exp(c\varphi)$. Its limit cases are:

- —a *logarithmic* (Archimedean) *spiral*: a planar curve ($z = 0$) that is described (in polar coordinates) by the equation $\rho = R_0 \exp(c\varphi)$;
- —a *cylindrical spiral*, which is described (in appropriate coordinates) by the equations $z = k\phi$, $\rho = R_0$;
- —a *circle* ($z = 0$, $\rho = R_0$);
- —a *semiline* (*ray*); and
- —a *straight line*.

2: Possible 2D orbits include:

- —a *plane*;
- —a *semiplane*;
- —a *sphere*;
- —a *circular cone*;
- —a *circular cylinder*; and
- —a *logarithmic cylinder*, i.e., a cylinder based on a logarithmic spiral.

4.3.2. Possible Orbits Are Exactly Possible Shapes

Comparing these orbits (and ellipsoids, the ultimate stable shapes) with astroshapes enumerated, e.g., in Vorontsov-Veliaminov (1987) we conclude that:

- First, our scheme describes all observed connected shapes.
- Second, all the above orbits, except the logarithmic cylinder, have actually been observed as shapes of celestial bodies.

For example, according to Chapter III of Vorontsov-Veliaminov (1987), galaxies consist of components of the following geometric shapes:

- *bars* (cylinders);
- *disks* (parts of the plane);
- *rings* (circles);
- *arcs* (parts of circles and lines);
- *radial rays*;
- *logarithmic spirals*;
- *spheres*; and
- *ellipsoids.*

It is easy to explain why a logarithmic cylinder has never been observed: from whatever point we view it, the logarithmic cylinder blocks all the sky, so it does not lead to any visible shape in the sky at all. With this explanation, we can conclude that we have a *perfect explanation of all observed astroshapes*.

4.3.3. Comment: We Can Also Explain Difficult-to-Explain Disconnected Shapes

In the above description, we only considered connected *continuous* subgroups $G_0 \subseteq G$. Connected continuous subgroups explain connected shapes.

It is natural to consider disconnected (in particular, discrete) subgroups as well; the orbits of these subgroups leads to disconnected shapes. Thus, we can explain these shapes, most of which modern astrophysics finds pathological and difficult to explain (see, e.g., Vorontsov-Veliaminov, 1987, Section I.3). For example, an orbit O of a discrete subgroup G_0' of the 1D group G_0 (whose orbit is a logarithmic spiral) consists of points whose distances r_n to the center forms a geometric progression: $r_n = r_0 \cdot k^n$. Such a dependence (called the Titzius–Bode law) has indeed been observed (as early as the 18th century) for planets of the solar system and for the satellites of the planets (this law actually led to the prediction and discovery of what are now called asteroids). Thus, we get a *purely geometric explanation of the Titzius–Bode law.*

Less-known examples of disconnected shapes that can be explained in this manner include:

- several parallel equidistant lines (Vorontsov-Veliaminov, 1987, Section I.3);

- several circles located on the same cone, whose distances from the cone's vertex form a geometric progression (Vorontsov-Veliaminov, 1987, Section III.9);
- equidistant points on a straight line (Vorontsov-Veliaminov, 1987, Sections VII.3 and IX.3);
- "piecewise circles": equidistant points on a circle; an example is MCG 0-9-15 (Vorontsov-Veliaminov, 1987, Section VII.3);
- "piecewise spirals": points on a logarithmic spiral whose distances from a center form a geometric progression: some galaxies of Sc type are like this (Vorontsov-Veliaminov, 1987).

Comment. Arnold has shown (see, e.g., Arnold, 1978; Thom, 1975) that dynamical systems theory explains why the observed shape should be *topological homeomorphic* to a spiral. We have explained even more: not only that this shape is *homeomorphic* to the spiral, but that, geometrically, this shape is *exactly a logarithmic spiral.*

4.3.4. This Idea Also Explains Evolution of Geometric Shapes, Their Relative Frequency, Directions of Rotation and of Magnetic Field

Our main idea can be used to explain not only the shapes themselves, but also how they evolve, which shapes are more frequent, etc. [For details, see, e.g., Kreinovich (1981), Kosheleva *et al.* (1982), Kosheleva and Kreinovich (1989), and Finkelstein *et al.* (1996)]. This explanation is, however, still on the physical level, so we still need to describe it in precise mathematical terms.

5. PROOF OF THE THEOREM

Since the criterion $\preccurlyeq$ is final, there exists one and only one optimal family of sets. Let us denote this family by A_{opt}.

1. Let us first show that this family A_{opt} is indeed G-invariant, i.e., that $g(A_{\text{opt}}) = A_{\text{opt}}$ for every transformation $g \in G$.

Indeed, let $g \in G$. From the optimality of A_{opt}, we conclude that for every $B \in \mathcal{A}$, $g^{-1}(B) \preccurlyeq A_{\text{opt}}$. From the G-invariance of the optimality criterion, we can now conclude that $B \preccurlyeq g(A_{\text{opt}})$. This is true for all $B \in \mathcal{A}$ and therefore the family $g(A_{\text{opt}})$ is optimal. But since the criterion is final, there is only one optimal family; hence, $g(A_{\text{opt}}) = A_{\text{opt}}$. So, A_{opt} is indeed invariant.

2. Let us now show an arbitrary set X_0 from the optimal family A_{opt} consists of orbits of $\geq(d - r)$-dimensional subgroups of the group G.

Indeed, the fact that A_{opt} is G-invariant means, in particular, that for every $g \in G$, the set $g(X_0)$ also belongs to A_{opt}. Thus, we have a (smooth) mapping $g \to g(X_0)$ from the d-dimensional manifold G into the $\leq r$-dimensional set $G(X_0) = \{g(X_0) | g \in G\} \subseteq A_{opt}$. In the following, we will denote this mapping by g_0.

Since $r < d$, this mapping cannot be 1–1, i.e., for some sets $X = g'(X_0) \in G(X_0)$, the preimage $g_0^{-1}(X) = \{g | g(X_0) = g'(X_0)\}$ consists of more than one point. By definition of $g(X)$, we can conclude that $g(X_0) = g'(X_0)$ iff $(g')^{-1}g(X_0) = X_0$. Thus, this preimage is equal to $\{g | (g')^{-1}g(X_0) = X_0\}$. If we denote $(g')^{-1}g$ by $\tilde{g}$, we conclude that $g = g'\tilde{g}$ and that the preimage $g_0^{-1}(X) = g_0^{-1}(g'(X_0))$ is equal to $\{g'\tilde{g} | \tilde{g}(X_0) = X_0\}$, i.e., to the result of applying g' to $\{\tilde{g} | \tilde{g}(X_0) = X_0\} = g_0^{-1}(X_0)$. Thus, each preimage ($g_0^{-1}(X) = g_0^{-1}(g'(X_0))$) can be obtained from one of these preimages [namely, from $g_0^{-1}(X_0)$] by a smooth invertible transformation g'. Thus, all preimages have the same dimension D.

We thus have a *stratification* (fiber bundle) of a d-dimensional manifold G into D-dimensional strata, with the dimension D_f of the factor-space being $\le r$. Thus, $d = D + D_f$, and from $D_f \le r$, we conclude that $D = d - D_f \ge n - r$.

So, for every set $X_0 \in A_{opt}$, we have a $D \ge (n - r)$-dimensional subset $G_0 \subseteq G$ that leaves X_0 invariant [i.e., for which $g(X_0) = X_0$ for all $g \in G_0$]. It is easy to check that if $g, g' \in G_0$, then $gg' \in G_0$ and $g^{-1} \in G_0$, i.e., that G_0 is a *subgroup* of the group G. From the definition of G_0 as $\{g | g(X_0) = X_0\}$ and the fact that $g(X_0)$ is defined by a smooth transformation, we conclude that G_0 is a smooth submanifold of G, i.e., a $\ge(n - r)$-dimensional subgroup of G.

To complete our proof, we must show that the set X_0 is a union of orbits of the group G_0. Indeed, the fact that $g(X_0) = X_0$ means that for every $x \in X_0$ and for every $g \in G_0$, the element $g(x)$ also belongs to X_0. Thus, for every element x of the set X_0, its entire orbit $\{g(x) | g \in G_0\}$ is contained in X_0. Thus, X_0 is indeed the union of orbits of G_0. QED

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